Baking integers out of π

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π Day

Here's a miracle!

transcendental

Some numbers are „almost integers":

$$e^{\pi} - \pi = 19{,}99909997\ldots$$

$$e + \pi + e^{\pi} + e^{\pi} + \pi^{e} = 59{,}99945\ldots$$

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$$e^{\pi} - \pi = 19{,}99909997\ldots$$

$$e + \pi + e^{\pi} + e^{\pi} + \pi^{e} = 59{,}99945\ldots$$

$$e^{\pi\sqrt{43}} = 12^3(9^2-1)^3 + 744 - 2{,}225\ldots \cdot 10^{-4}$$

$$e^{\pi\sqrt{67}} = 12^3(21^2-1)^3 + 744 - 1{,}337\ldots \cdot 10^{-6}$$

$$e^{\pi\sqrt{163}} = 12^3(231^2-1)^3 + 744 - 7{,}499\ldots \cdot 10^{-13}$$

Ramanujan's constant

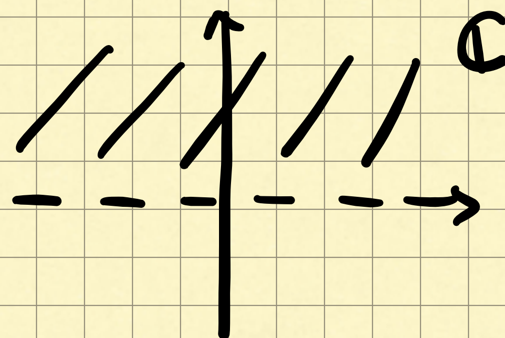
&

TOPIC OF TODAY!

Have you heard about j-invariant?

famous elliptic modular function!

j is defined on $\mathbb{H}$:



Its „q-expansion":

$$j(\tau) = q^{-1} + 744 + 196884\,q + 21493760\,q^2 + \ldots$$

where $q = e^{2\pi i \cdot \tau}$.

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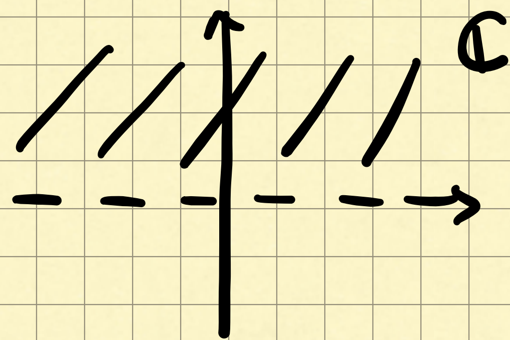

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where $q = e^{2\pi i \cdot \tau}$.

We choose $\tau := \dfrac{1+\sqrt{-163}}{2}$.

Since $q = e^{2\pi i \cdot \tau} = -e^{-\pi\sqrt{163}}$ is tiny,

$$e^{\pi \cdot \sqrt{163}} = -q^{-1} \approx 744 - j(\tau).$$

Claim follows from:

$j(\tau)$ is an integer!

But what are values of j?...

They are fascinating!

Say, $|\mathbb{Q}[z] : \mathbb{Q}|$ = deg

imaginary algebraic number

Dear deg!
Are you ?

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$\boxed{\text{Dear deg! Are you?}}$

≥ 3 ↙ $j(\tau)$ is transcendental

$= 2$ ↘ $j(\tau)$ is algebraic & can be expressed in radicals!

And values are often cubes:

$$j(\sqrt{-5}) = \left(2\cdot\sqrt{5}\,(13+5\sqrt{5})\right)^3$$

$$j\left(\frac{1+\sqrt{-163}}{2}\right) = -640320^3$$

So what's special about 163?

Heegner numbers:

1, 2, 3, 7, 11, 19, 43, 67, 163.

All the numbers $d > 0$ s.t.

$\mathbb{Q}[\sqrt{-d}]$ has unique factorization!

conjectured by Gauss :)

In fancy language:

$\mathrm{Cl}(\mathbb{Q}[\sqrt{-d}])$ has size 1.

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In fancy language:
$Cl(\mathbb{Q}[\sqrt{-d}])$ has size 1.

Hard theorem: $\forall d = 4k+3$, $\tau = \frac{1+\sqrt{-d}}{2}$
$j(\tau)$ is algebraic integer of degree $|Cl(\mathbb{Q}[\sqrt{-d}])|$ over $\mathbb{Q}[\sqrt{-d}]$.

: $|Cl| = 1 \Rightarrow j(\tau) \in \mathbb{Q}[\sqrt{-d}] \cap \mathbb{R} \Rightarrow$

theorem q-expansion

$j(\tau) \in \mathbb{Q}$ and is alg integer $\Rightarrow j(\tau) \in \mathbb{Z}$!

And what is Cl? ideal class group

measures (non-)unique factorization

K number field

$\mathcal{O}_K$ algebraic integers $\in K$

(roots of monic polynomials in $\mathbb{Z}[t]$)

fractional

$$Cl(K) := \mathcal{J}_K / P_K \qquad I \subseteq \mathcal{O}_K$$

principal fractional

finite group

fractional: $I = r^{-1}J$, J ideal

"ideal"

unique factorization $\Rightarrow$ $\mathcal{J}_K = P_K$

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(roots of monic polynomials in $\mathbb{Z}[t]$)

$$Cl(K) := J_K / P_K$$

J_K: fractional $I \subseteq \mathcal{O}_K$

P_K: principal fractional

finite group

fractional "ideal": $I = r^{-1} J$, J ideal

unique factorization $\Rightarrow J_K = P_K$

Example: $K = \mathbb{Q}(\sqrt{-5})$

$I = (2, 1+\sqrt{-5})$ is NOT principal!

$I^2 = (2) \quad \rightsquigarrow \quad Cl(K) = \mathbb{Z}/2\mathbb{Z}$

related to: $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$

♥ „ideal numbers" (Kummer) $\rightsquigarrow$ non-principal ideals (Dedekind)

⑥

– So tell me, what is j?

– Well...

Simplest f: $f(\tau) = f(\tau+1)$

is $f(\tau) := e^{2\pi i \tau}$

And simplest f: $f(\tau) = f(\tau+1) = f(-\frac{1}{\tau})$

$SL_2(\mathbb{Z})$ - invariant & holomorphic

is $f(\tau) := j(\tau) =$

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is $f(\tau) := j(\tau) =$ $SL_2(\mathbb{Z})$-invariant & holomorphic

$= \boxed{\text{~~horrible formula~~}}$

Cool fact: j parametrizes elliptic curves / $\mathbb{C}$! (and $\mathbb{Z}$)

Because: $E = \mathbb{C}/\Lambda$, $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$

depends on $\omega_2/\omega_1 \in \mathbb{H}$ $\rightsquigarrow$

$Ell_{\mathbb{C}} \simeq \mathbb{H}/SL_2(\mathbb{Z}) \xrightarrow[\sim]{j} \mathbb{C}$, so j is the coordinate.

7

And why did you say that

Hard theorem: $\forall d = 4k+3$, $\tau = \frac{1+\sqrt{-d}}{2}$

$j(\tau)$ is algebraic (integer)

of degree $|Cl(\mathbb{Q}[\sqrt{-d}])|$ over $\underset{\substack{\| \\ K}}{\mathbb{Q}[\sqrt{-d}]}$?

Claim: $Cl(K) \simeq Gal\left(K[j(\tau)]/K\right)$

$\forall$ such τ, K

Claim $\Rightarrow$ Gal finite & abelian

$\Rightarrow j(\tau)$ expressed in radicals!

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Proof idea: $\mathcal{E} := \left\{ \begin{matrix} E/\mathbb{C} \text{ ell curve} \\ End(E) \simeq \mathcal{O}_K \end{matrix} \right\}$ with complex multiplication

- $Gal \circlearrowright \left\{ \begin{matrix} j(\tau) \text{ and its} \\ \text{conjugates} \\ \text{all } j(\tau') \in j(\mathcal{E}) \end{matrix} \right\} \rightsquigarrow Gal \circlearrowright \mathcal{E}$
- $Cl \simeq \mathcal{E}$ (exercise) $\rightsquigarrow Cl \circlearrowright \mathcal{E}$

AGREE! ...

and the degree is correct! :)

Recall:

$$j(\tau) = q^{-1} + 744 + 196884\,q + 21493760\,q^2 + \dots$$

What's special about 196884?

It is breath of the Monster!

Finite simple groups:
- classification (~12000 pages)
- 18 infinite families (C_p, A_n etc)
- 26 groups don't fit in
- Monster group: the biggest!

(2^{nd}: Baby Monster, others: Happy Family)
- $|\text{Monster}| \approx 8 \cdot 10^{53}$

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Monstrous Moonshine:

$$\dim \begin{pmatrix} \text{smallest} \\ \text{irreducible} \\ \text{representation} \\ \text{of Monster} \end{pmatrix} = 196883$$

and other coeffs are linear combinations with small coeffs >0 of dims of other reps... Borcherds!